Similar Solutions of Gas Pressure Within Semi-infinite Capillary Tubes

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The similar solutions of Ducoffe's equation are presented. Through use of the solutions, the time variation of the gas pressure within thin semi-infinite capillary tubes is solved when the pressure at one end is changed discontinuously from an initial uniform value to a new constant one. The rate of pressure variation propagation decreases as the initial tube pressure becomes lower, and a finite velocity pressure variation front is formed if the initial tube pressure is a vacuum. Very close to the open end, the order of the propagation rate is reversed.

Introduction

UNSTEADY variation of the gas pressure within thin capillary tubes is approximated by

$$\frac{\partial P}{\partial T}=\frac{R^2}{8\mu}\frac{\partial(P\partial P/\partial Z)}{\partial Z} \tag{1}$$

where P is the pressure, T the time, Z the length measured along the tubes, R the radius of the tubes, and μ the viscosity coefficient of the gases.

Equation (1) was derived by Ducoffe[1] on a crude assumption of local incompressible Poiseulle flow; the present author has demonstrated[2] its formal derivation from the compressible Navier-Stokes equations by assuming laminar subsonic isothermal viscous flow of ideal gases within tubes of small radius-to-length ratio. By use of Ducoffe's equation, the transient pressure variation along capillary tubes of finite length with one end attached to pressure transducers having finite volume cavities and with other end open to constant pressure fields has been studied numerically and compared with experiments in connection with wind-tunnel pressure measurements.[1,2]

If the length of the tubes is infinite, the flow within the tube has no reference length along the axis and, therefore, Eq. (1) must have similar solutions. The lag time in the pressure measurements is heavily affected by ratio of the transducer cavity volume to the tube inside volume. Thus, it is impossible to estimate the lag time solely from the tube geometry, but these similar solutions help our basic understanding of the nature of lag time buildup in thin, long tubes.

In this paper, the similar solutions of Ducoffe's equation (1) are presented and through use of these solutions the gas pressure variation within a semi-infinite capillary tube whose pressure at one end is changed discontinuously from an initial uniform value P_∞ to a new constant pressure P_0 is studied.

Similar Solutions

Equation (1) can be rewritten with nondimensional values as

$$\frac{\partial p}{\partial t}=\frac{1}{2}\left(\frac{\partial^2 p^2}{\partial z^2}\right) \tag{2}$$

where

$$p=P/P_{\rm ref},\quad z=Z/R,\quad t=TP_{\rm ref}/8\mu \tag{3}$$

and $P_{\rm ref}$ is an arbitrarily chosen reference pressure.

For $z\neq 0$ the initial condition is

$$p(0,z)=p_\infty\,(=P_\infty/P_{\rm ref}\text{: const}) \tag{4}$$

and the boundary conditions are

$$p(t,0)=p_0\,(=P_0/P_{\rm ref}\text{: const})$$

and

$$p(t,\infty)=p_\infty \tag{5}$$

The similar solutions of Eq. (2) including $p(t,z)=\text{const}$ as a special case can be obtained by introducing a transformation

$$\eta=zt^{-1/2} \tag{6}$$

which reduces Eq. (2) to

$$\frac{d^2f^2}{d\eta^2}+\frac{\eta df}{d\eta}=0 \tag{7}$$

or, equivalently,

$$2f\left(\frac{d^2f}{d\eta^2}\right)+\left(\frac{2df}{d\eta}+\eta\right)\frac{df}{d\eta}=0 \tag{8}$$

where

$$f(\eta)=p(t,z) \tag{9}$$

The initial and the boundary conditions [Eqs. (4) and (5)] become

$$f(0)=p_0 \text{ and } f(\infty)=p_\infty \tag{10}$$

respectively.

If P_0 is nonzero, the reference pressure $P_{\rm ref}$ can be taken equal to P_0 and Eq. (10) becomes

$$f(0)=1 \text{ and } f(\infty)=p_\infty \tag{11}$$

In the case of $P_0=0$, another choice of $P_{\rm ref}=P_\infty$ can be made and Eq. (10) becomes

$$f(0)=0 \text{ and } f(\infty)=1 \tag{12a}$$

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Numerical Integrations

The ordinary differential equation (8) is integrated numerically by means of the Runge-Kutta-Gill method.[3] In the case of the boundary conditions (11) the integration is started from $\eta=0$ by assuming $f'(0)$, and $f'(0)$ is modified iteratively until the prescribed value of $f(\infty)$ is obtained. This iterative shooting procedure converges rapidly because, as long as $f(\eta)$ remains positive, $f(\infty)$ is finite for any $f'(0)$, and at large η the derivative $f'(\eta)$ decreases monotonically to zero as η increases.

If the solution of Eq. (8) has a point where $f(\eta)$ vanishes, this point is a singular one, since the coefficient of the highest derivative in Eq. (8) vanishes. Let η_0 be a point where $f(\eta_0)$ vanishes and, furthermore, $f(\eta)>0$ for $0\leq\eta<\eta_0$; then from Eq. (8)

$$f'(\eta_0)\,(2f'(\eta_0)+\eta_0)=0$$

Therefore,

$$f'(\eta_0)=0 \quad \text{or} \quad -\eta_0/2 \tag{13}$$

must hold as long as $f''(\eta_0)$ is finite. If $f(0)>0$ and $f(\infty)=0$ are the boundary conditions, the solution has a discontinuous first derivative

$$f'(\eta_0)=\begin{cases} -\eta_0/2 & \text{as} \quad \eta\to\eta_0-0 \\ 0 & \text{as} \quad \eta\to\eta_0+0 \end{cases} \tag{14}$$

at $\eta=\eta_0$. From this point to infinity, $f(\eta)$ remains zero. This singular solution of Eq. (8) can be obtained as follows: If $f(\eta)$ is expanded around η_0 for $\eta\leq\eta_0$ as

$$f(\eta)=\sum_{n=0}^{\infty} a_n(\eta-\eta_0)^n \tag{15}$$

substitution of Eq. (15) into Eq. (8) yields

$$(2a_1^2+a_1\eta_0)+(12a_1a_2+a_1+2a_2\eta_0)(\eta-\eta_0)$$
$$+(24a_1a_3+12a_2^2+2a_2+3a_3\eta_0)(\eta-\eta_0)^2$$
$$+O(\eta-\eta_0)^3=0$$

From the conditions of the vanishing of the coefficients of the terms $(\eta-\eta_0)^n$, $(n=0, 1,\ldots)$

$$\begin{aligned} f'(\eta_0) &\equiv a_1=-\eta_0/2 \\ f''(\eta_0) &\equiv 2a_2=-1/4 \end{aligned} \tag{16}$$

must hold at $\eta=\eta_0-0$. With an arbitrarily chosen positive η_0, Eq. (8) can be integrated numerically from $\eta=\eta_0$ to $\eta=0$ with Eq. (16) and $f(\eta_0)=0$ as the initial values; the resulting $f(\eta)$, $(0\leq\eta\leq\eta_0)$ be transformed affinely to $g(\zeta)$ by

$$g(\zeta)=f(\eta)/A$$

and (17)

$$\zeta=\eta/\sqrt{A}$$

From Eqs. (8) and (17)

$$\frac{1}{A}\left(\frac{d^2g^2}{d\zeta^2}+\frac{\zeta dg}{d\zeta}\right)=0 \tag{18}$$

which is exactly the same as Eq. (8). $g(\zeta)$ satisfies the boundary conditions

$$g(0)=f(0)/A \quad \text{and} \quad g(\eta_0/\sqrt{A})=0 \tag{19}$$

Therefore, the singular solution of Eqs. (8) and (11) can be obtained by putting $A=f(0)$ and by replacing g with f and ζ with η. The coordinate of the singular point is found to be $\eta_0=1.616125$--.

The boundary condition [Eq. (12)] is another singular case and needs special treatment. At the singular point $\eta=0$, $f'(0)=0$ and $f''(0)=-1/3$ result from Eq. (8) if $f''(0)$ and $f'''(0)$ are finite. But these will lead to an unreasonably negative $f(\eta)$ near $\eta=0$ and, for this reason, $f'(0)$ and $f''(0)$ must be infinite. Let $f(\eta)$ be approximated near $\eta=0$ by

$$f(\eta)=C\eta^a[1+O(\eta^a)], \quad (C\neq 0,\ 0<a<1) \tag{20}$$

Substitution of Eq. (20) into Eq. (8) yields

$$C^2a(2a-1)\eta^{2(a-1)}[1+O(\eta^a)]=0$$

From this, $a=1/2$ is obtained. After some manipulation, the series expansion of $f(\eta)$ around $\eta=0$ in terms of $\eta^{1/2}$ results in

$$f(\eta)=C\eta^{1/2}-(1/15)\eta^2+(1/300C)\eta^{7/2}-(1/74250C^2)\eta^5$$
$$-(61/16{,}632{,}000C^3)\eta^{13/2}+(163/1{,}060{,}290{,}000C^4)\eta^8$$
$$+O(\eta^{19/2}) \tag{21}$$

The solution near $\eta=0$ is assumed to be obtained by Eq. (21) and the initial value of Eq. (8) is calculated at some small positive η. From that point on, Eq. (8) is integrated numerically. The constant C is initially put equal to unity and the integration yields $f(\infty)=1.31205$--; the affine transformation of Eq. (17) with $A=f(\infty)$ leads to Eq. (18) and the boundary conditions similar to those of Eq. (12a)

$$g(0)=0 \quad \text{and} \quad g(\infty)=1 \tag{12b}$$

This solution can also be obtained directly from Eqs. (8) and (21) with $C=A^{3/4}=1.22592$--.

Results and Discussion

The results of integration for the boundary conditions [Eq. (11)] are presented on Figs. 1-3. In Fig. 1 the distributions of $f(\eta)$ normalized by $f(0)-f(\infty)$ are plotted: the rate of the pressure variation propagation is greater for higher P_∞, but the order is reversed very close to the open ends of the tubes. The velocity of the pressure propagation fronts is, in general, infinite since the basic equation (1) is of the parabolic type; but for a particular case of $f(\infty)=0$, the effect of sudden change of pressure at $\eta=0$ propagates at finite velocity and

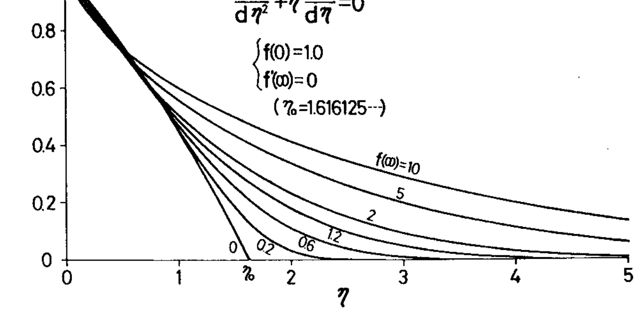

Fig. 1 Normalized pressure $(f(\eta)-f(\infty))/(f(0)-f(\infty))$ vs η for $f(0)=1.0$ and $f'(\infty)=0$.

Table 1 First derivatives of the similar solutions at $\eta = 0$

$f(0) = 0.0$	1.0	1.0	1.0	1.0	1.0	1.0
$f(\infty) = 1.0$	0.0	0.1	0.2	0.4	0.6	0.8
$f'(0) = \infty$	−0.4437	−0.4126	−0.3777	−0.2985	−0.2084	−0.1086
$f(0) = 1.0$	1.0	1.0	1.0	1.0	1.0	1.0
$f(\infty) = 1.2$	1.4	1.6	1.8	2.0	5.0	10.0
$f'(0) = 0.1169$	0.2414	0.3733	0.5121	0.6575	3.5034	10.3421

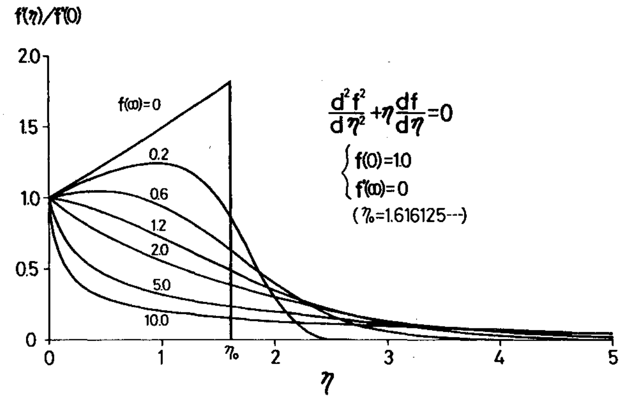

Fig. 2 Normalized first derivatives of pressure $f'(\eta)/f'(0)$ vs η for $f(0) = 1.0$ and $f'(\infty) = 0$.

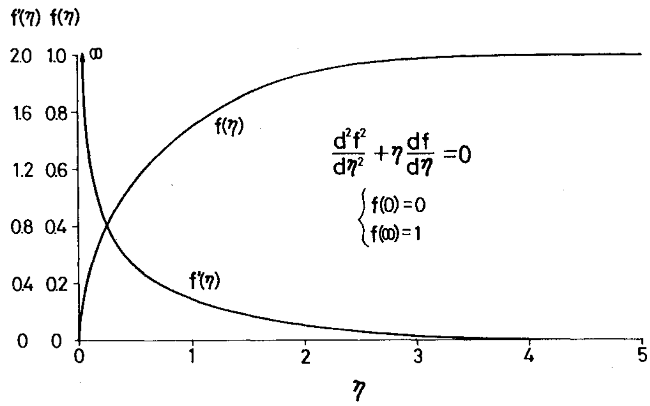

Fig. 4 A singular solution for vacuum open end pressure.

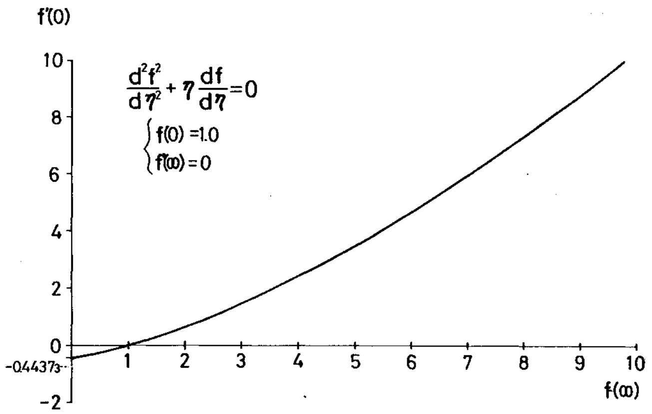

Fig. 3 First derivative of pressure at $\eta = 0$, $f'(0)$ vs pressure at infinity $f(\infty)$.

the effect never reaches beyond $\eta = \eta_0$. The velocity of the front is, from Eq. (6),

$$\frac{dz}{dt} = \frac{\eta_0}{2\sqrt{t}} \tag{22}$$

which is decreasing as time elapses. These conclusions have already been pointed out by the present author in an earlier paper[2] through finite difference numerical analysis and experiments with thin capillary tubes. The present analysis shows that these finite velocity shock-wave-like fronts are strictly formed only for pressure propagating into a vacuum. This conclusion corresponds exactly to the one proposed by Zel'dovich and Kompaneets[4] for nonlinear heat conduction equations. The same type of solution has been found by Wagner[5] for nonlinear diffusion equations.†

Ducoffe's equation (1) is based on the assumptions of low mean velocity and local incompressible flow within the tubes. Because of these assumptions, the sound velocity never comes in the analysis. Therefore, the velocity [Eq. (22)] is valid only for the pressure front speeds lower than that of sound.

Figure 2 shows the distributions of $f'(\eta)$ normalized by $f'(0)$. The value of $f'(0)$ is important for the integration of Eq. (8) and is shown in Fig. 3 as well as in Table 1.

The solution for the boundary conditions of Eq. (12) is plotted in Fig. 4 with its first derivative. The function $f(\eta)$ approaches tangentially to the vertical axis at $\eta = 0$ and $f'(\eta)$ is positive infinite at this point.

Conclusions

The similar solutions of Ducoffe's equation are presented and the time variation of the gas pressure within thin semi-infinite capillary tubes is solved through the similar solutions, with the assumptions of low mean velocity and an instantaneous pressure change at one open end of the tubes. Under these assumptions, the effect of a pressure change at one end propagates at infinite velocity into the tubes unless the initial tube pressure is a vacuum. For this particular case, the pressure propagation front has a finite velocity.

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†The author appreciates the notification of this point by one of the reviewers.